

# OPTIMAL ESTIMATES FOR HARMONIC FUNCTIONS IN THE UNIT BALL

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ABSTRACT. We find the sharp constants  $C_p$  and the sharp functions  $C_p = C_p(x)$  in the inequality

$$|u(x)| \leq \frac{C_p}{(1 - |x|^2)^{(n-1)/p}} \|u\|_{h^p(B^n)}, u \in h^p(B^n), x \in B^n,$$

in terms of Gauss hypergeometric and Euler functions. This extends and improves some results of Axler, Bourdon and Ramey ([2]), where they obtained similar results which are sharp only in the cases  $p = 2$  and  $p = 1$ .

## 1. INTRODUCTION AND STATEMENT OF THE RESULTS

Let  $n \geq 2$  and let  $h^p(B^n)$ ,  $1 \leq p \leq \infty$ , be the harmonic Hardy spaces on the unit  $n$ -dimensional ball  $B^n$  in the Euclidean space  $\mathbf{R}^n$ , the space of all harmonic functions  $u$  satisfying growth condition

$$\|u\|_p^p := \|u\|_{h^p(B^n)}^p = \sup_{0 < r < 1} \int_S |u(r\zeta)|^p d\sigma(\zeta) < \infty$$

where  $S = S^{n-1}$  is the unit sphere and  $\sigma$  is the unique normalized rotation invariant Borel measure on  $S$ . It is well known that a harmonic function  $u \in h^p(B^n)$  posses radial (angular) limit  $u(\zeta)$  in almost all points on sphere  $\zeta \in S^{n-1}$  and that for  $p > 1$  it is possible to express in the form

$$(1.1) \quad u(x) = \int_S P(x, \zeta) u(\zeta) d\sigma(\zeta),$$

where

$$P(x, \zeta) = \frac{1 - |x|^2}{|x - \zeta|^n}, \zeta \in S$$

is Poisson kernel.

The maximum principle implies that, if  $u \in h^\infty(B^n)$ , then  $|u(x)| \leq \|u\|_\infty$ . On the other hand, it follows from the Poisson representation formula (1.1) that, if  $u \in h^1(B^n)$ , then

$$|u(x)| \leq \sup_{\zeta \in S} P(x, \zeta) \|u\|_1.$$

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Then

$$\sup_{\zeta \in S} P(x, \zeta) = \frac{(1 + |x|)^n}{(1 - |x|^2)^{n-1}}.$$

In this work we find a representation for the sharp constants  $C_p$  and the sharp functions  $C_p = C_p(x)$  in the inequality

$$|u(x)| \leq \frac{C_p}{(1 - |x|^2)^{(n-1)/p}} \|u\|_p$$

where  $x$  is an arbitrary point in the unit ball  $B^n$ .

It is well known that  $C_p(x)$  is a bounded function in  $B^n$  for  $1 \leq p \leq \infty$ , and the power  $(n-1)/p$  is optimal. See [2, Proposition 6.16] for the case  $n \geq 2$  and  $1 \leq p \leq \infty$  and [9, Lemma 5.1.1] for the case of analytic functions ( $n = 2$ ) and  $0 < p < \infty$ . In the case when  $h^p(B^n)$  is Hilbert space, that is for  $p = q = 2$  we have next sharp point estimate

$$(1.2) \quad |u(x)| \leq \sqrt{\frac{1 + |x|^2}{(1 - |x|^2)^{n-1}}} \|u\|_{h^2(B^n)}.$$

The previous inequality is obtained in [2, Proposition 6.23] using the fact that  $h^2(B^n)$  is Hilbert space and Riesz representation theorem for functionals by use of the scalar product and Cauchy-Schwartz inequality in the general setting  $|\langle x, y \rangle| \leq \|x\| \|y\|$ .

Let  $q$  be as usual conjugate with  $p$  that is  $1/p + 1/q = 1$ . In this paper we generalize (1.2) by proving the following two theorems.

**Theorem 1.1.** *Let  $1 < p \leq \infty$ . For all  $u \in h^p(B^n)$  and  $x \in B^n$  we have the following sharp inequality*

$$(1.3) \quad |u(x)| \leq \frac{C_p(x)}{(1 - |x|^2)^{(n-1)/p}} \|u\|_{h^p(B^n)}$$

where

$$C_p(x) = (F(-1 + n - \frac{nq}{2}, \frac{1}{2}(n - nq), \frac{n}{2}, |x|^2))^{1/q},$$

and  $F$  is the Gauss Hypergeometric functions.

**Theorem 1.2.** *Let  $1 < p \leq \infty$ . For all  $u \in h^p(B^n)$  and  $x \in B^n$  we have the sharp inequality*

$$|u(x)| \leq \frac{C_p}{(1 - |x|^2)^{(n-1)/p}} \|u\|_{h^p(B^n)}$$

where

$$C_p = \begin{cases} 1, & \text{if } q \leq 2 - \frac{2}{n}; \\ \left( \frac{2^{nq-n} \Gamma(\frac{n}{2}) \Gamma(\frac{1+nq-n}{2})}{\sqrt{\pi} \Gamma(\frac{nq}{2})} \right)^{1/q}, & \text{if } q > 2 - \frac{2}{n}. \end{cases}$$

*Remark 1.3.* The cases  $p = \infty$  and  $p = 1$  are already considered in the introduction of this paper and are well-known. For the case  $p = \infty$  i.e.

$q = 1$  we have  $C_p(x) = 1$ . For the case  $p = 1$  we have  $C_1(x) = (1 + |x|)^n$ . We will assume in the sequel that  $1 < q < \infty$ . For  $p = q = 2$ ,

$$F(-1 + n - \frac{nq}{2}, \frac{1}{2}(n - nq), \frac{n}{2}, |x|^2) = 1 + |x|^2,$$

thus (1.3) coincides with (1.2).

*Remark 1.4.* If instead of  $|u(x)|$  in Theorems 1.1 and 1.2, we put the norm of its gradient  $|\nabla u(x)|$ , then, instead of  $(n-1)/p$  we have the power  $1+(n-1)/p$ . See [4], [5], [7], [6] and [8] for related results.

## 2. REPRESENTATIONS FOR $C_p(x)$ AS SINGLE INTEGRALS

If  $u$  is harmonic, and  $T$  is an orthogonal transformation, then  $u \circ T$  is a harmonic function. Using this fact and  $\|u \circ T\|_{h^p(B^n)} = \|u\|_{h^p(B^n)}$  it is easy to see that  $C_p(x) = C_p(r_x)$  where  $r_x = (|x|, 0, \dots, 0)$  is the vector on the  $e_1$  axis of the same norm as  $x$ .

In the sequel we will use Möbius transform of the multidimensional ball. Let us recall some basic facts from [1]. In general a Möbius transform  $T_x : B^n \rightarrow B^n$  has form

$$T_x(y) = \frac{(1 - |x|^2)(y - x) - |y - x|^2 x}{[y, x]^2}, \quad y \in B^n$$

where  $[y, x] = |y||x - y^*|$ ,  $y^* = y/|y|^2$ . In special case if  $x = r_x$  we have

$$T_{r_x}(y) = (1 - |x|^2) \frac{y - r_x}{|y - r_x|^2} - r_x.$$

Jacobi determinant of the  $T_{r_x} : S \rightarrow S$  in the point  $\eta \in S$  is

$$J_{T_{r_x}}(\eta) = \left( \frac{1 - |x|^2}{|\eta - r_x|^2} \right)^{n-1}.$$

By applying Holder inequality in the relation

$$u(x) = \int_S P(x, \zeta) u(\zeta) d\sigma(\zeta),$$

we have

$$|u(x)| \leq \left( \int_S P^q(x, \zeta) d\sigma(\zeta) \right)^{1/q} \|u\|_p.$$

Let

$$I_q = \int_S P^q(x, \zeta) d\sigma(\zeta).$$

In the integral we make change of variables  $\zeta = -T_{r_x}(\eta)$ , where

$$T_{r_x}(\eta) = (1 - |x|^2) \frac{\eta - r_x}{|\eta - r_x|^2} - r_x$$

is Möbius transform of the unit ball  $B^n$ . Then

$$|r_x - \zeta| = \frac{1 - |x|^2}{|\eta - r_x|}$$

and

$$d\sigma(\zeta) = \left( \frac{1 - |x|^2}{|\eta - r_x|^2} \right)^{n-1} d\sigma(\eta).$$

So

$$\begin{aligned} I_q &= \int_S \frac{(1 - |x|^2)^q}{\frac{(1 - |x|^2)^{nq}}{|r_x - \eta|^{nq}}} \left( \frac{1 - |x|^2}{|\eta - r_x|^2} \right)^{n-1} d\sigma(\eta) \\ &= \int_S (1 - |x|^2)^{q-nq+n-1} |\eta - r_x|^{nq-2n+2} d\sigma(\eta). \end{aligned}$$

Further

$$I_q^{1/q} = \frac{1}{(1 - |x|^2)^{(n-1)/p}} \left( \int_S |\eta - r_x|^{nq-2n+2} d\sigma(\eta) \right)^{1/q}$$

and

$$|u(x)| \leq \frac{1}{(1 - |x|^2)^{(n-1)/p}} \left( \int_S |\eta - r_x|^{nq-2n+2} d\sigma(\eta) \right)^{1/q} \|u\|_{h^p(B^n)},$$

or

$$|u(x)| \leq \frac{C_p(x)}{(1 - |x|^2)^{(n-1)/p}} \|u\|_{h^p(B^n)},$$

where

$$C_p(x) = \left( \int_S |\eta - r_x|^{nq-2n+2} d\sigma(\eta) \right)^{1/q}.$$

The sharp constant  $C_p$  is

$$C_p = \sup_{x \in B^n} \left( \int_S |\eta - r_x|^{nq-2n+2} d\sigma(\eta) \right)^{1/q}.$$

For  $n = 2$  we have

$$\begin{aligned} C_p^q(x) &= \int_{S^1} |\eta - r_x|^{2q-2} d\sigma(\eta) = \frac{1}{2\pi} \int_0^{2\pi} (1 + |x|^2 - 2|x| \cos \theta)^{q-1} d\theta \\ &= \frac{1}{\pi} \int_0^\pi (1 + |x|^2 - 2|x| \cos \theta)^{q-1} d\theta. \end{aligned}$$

Let  $n > 2$  and

$$K = \{(\theta_1, \dots, \theta_{n-2}, \varphi) : 0 \leq \theta_1, \dots, \theta_{n-2} \leq \pi, 0 \leq \varphi \leq 2\pi\}.$$

Using spherical coordinates  $(\eta_1, \dots, \eta_{n-1}, \eta_n) = (\theta_1, \dots, \theta_{n-2}, \varphi)$  we get

$$\begin{aligned} C_p^q(x) &= \int_S |\eta - r_x|^{nq-2n+2} d\sigma(\eta) \\ &= \frac{1}{\omega_{n-1}} \int_K (1 + |x|^2 - 2|x| \cos \theta_1)^{nq/2-n+1} \sin^{n-2} \theta_1 \dots \sin \theta_{n-2} d\theta_1 \dots d\varphi \\ &= \frac{2\pi}{\omega_{n-1}} I_n \int_0^\pi \sin^{n-2} \theta_1 (1 + |x|^2 - 2|x| \cos \theta_1)^{nq/2-n+1} d\theta_1, \end{aligned}$$

where

$$I_n = \int_0^\pi \sin^{n-3} \theta_2 d\theta_2 \dots \int_0^\pi \sin \theta_{n-2} d\theta_{n-2}$$

and  $\omega_{n-1}$  is volume of  $n - 1$ -dimensional unit sphere. Since

$$\int_0^\pi \sin^{n-2} \theta d\theta = \frac{\sqrt{\pi} \Gamma((n-1)/2)}{\Gamma(n/2)},$$

and

$$I_n = \frac{\omega_{n-1}}{2\pi \int_0^\pi \sin^{n-2} \theta d\theta}$$

we have

$$(2.1) \quad C_p^q(x) = \frac{\Gamma(n/2)}{\sqrt{\pi} \Gamma((n-1)/2)} \int_0^\pi \sin^{n-2} \theta (1 + |x|^2 - 2|x| \cos \theta)^{nq/2-n+1} d\theta.$$

Note that (2.1) is also true for  $n = 2$  since  $\Gamma(1/2) = \sqrt{\pi}$ .

### 3. REPRESENTATIONS FOR $C_p(x)$ AS A GAUSS HYPERGEOMETRIC FUNCTION AND THE PROOF OF THEOREM 1.1

We recall the classical definition of the Gauss hypergeometric function:

$$F(a, b, c, z) = 1 + \sum_{n=1}^{\infty} \frac{(a)_n (b)_n}{(c)_n n!} z^n,$$

where  $(d)_n = d(d+1) \cdots (d+n-1)$  is the Pochhammer symbol. The series converges at least for complex  $z \in \mathbf{U} := \{z : |z| < 1\} \subset \mathbf{C}$  and for  $z \in \mathbf{T} := \{z : |z| = 1\}$ , if  $c > a + b$ . Here  $\mathbf{C}$  is the complex plane. For  $\Re(c) > \Re(b) > 0$  we have the following well-known formula

$$(3.1) \quad F(a, b, c, z) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 \frac{t^{b-1} (1-t)^{c-b-1}}{(1-tz)^a} dt.$$

It is easy to check the following formula

$$(3.2) \quad \frac{d}{dz} F(a, b, c, z) = \frac{ab F(1+a, 1+b, 1+c, z)}{c}.$$

We will use Kummer's Quadratic Transformation of a hypergeometric function,

$$(3.3) \quad F(a, b, 2b, \frac{4z}{(1+z)^2}) = (1+z)^{2a} F(a, a + \frac{1}{2} - b, b + \frac{1}{2}, z^2).$$

**Lemma 3.1.** *For  $n \geq 2$ ,  $0 \leq r \leq 1$  and  $q \geq 1$  we have*

$$(3.4) \quad \begin{aligned} a_q(r) &:= \int_0^\pi \sin^{n-2} t (1 + r^2 - 2r \cos t)^{nq/2-n+1} dt \\ &= \frac{\sqrt{\pi} \Gamma(\frac{n-1}{2}) F(-1 + n - \frac{nq}{2}, \frac{1}{2}(n - nq), \frac{n}{2}, r^2)}{\Gamma(\frac{n}{2})}, \end{aligned}$$

where  $F$  is the Gauss hypergeometric function.

*Proof of Lemma 3.1.* First of all

$$1 + r^2 - 2r \cos t = (1+r)^2 \left(1 - z \frac{1 + \cos t}{2}\right),$$

where

$$z = \frac{4r}{(1+r)^2}.$$

By taking the substitution

$$u = \frac{1 + \cos t}{2}$$

we obtain

$$du = -\frac{1}{2} \sin t dt$$

and

$$\sin t = 2u^{\frac{1}{2}}(1-u)^{\frac{1}{2}}$$

and therefore

$$\begin{aligned} a_q(r) &= 2(1+r)^{nq-2n+2} \int_0^1 \sin^{n-3} t (1-zu)^{nq/2-n+1} du \\ &= 2^{n-2} (1+r)^{nq-2n+2} \int_0^1 \frac{u^{\frac{n-3}{2}} (1-u)^{\frac{n-3}{2}}}{(1-zu)^{n-1-nq/2}} du. \end{aligned}$$

By taking

$$a = n-1-nq/2, \quad b = \frac{n-1}{2} \text{ and } c = 2b = n-1$$

and by using the formula (3.1) and (3.3), we obtain

$$\begin{aligned} a_q(r) &= 2^{n-2} (1+r)^{-2a} \frac{\Gamma(b)\Gamma(c-b)}{\Gamma(c)} F(a, b, c, z) \\ &= 2^{n-2} (1+r)^{-2a} \frac{\Gamma(b)\Gamma(c-b)}{\Gamma(c)} F(a, b, 2b, \frac{4r}{(1+r)^2}) \\ &= 2^{n-2} \frac{\Gamma(b)\Gamma(c-b)}{\Gamma(c)} F(a, a + \frac{1}{2} - b, b + \frac{1}{2}, r^2). \end{aligned}$$

By using

$$2^{n-2} \frac{\Gamma(b)\Gamma(c-b)}{\Gamma(c)} = 2^{n-2} \frac{\Gamma(\frac{n-1}{2})\Gamma(\frac{n-1}{2})}{\Gamma(n-1)} = \sqrt{\pi} \frac{\Gamma(\frac{n-1}{2})}{\Gamma(\frac{n}{2})}$$

we obtain finally (3.4). □

**Lemma 3.2.** *Under the conditions of Lemma 3.1 we have*

$$a_q(s) = \begin{cases} \frac{\sqrt{\pi} \Gamma(\frac{n-1}{2})}{\Gamma(\frac{n}{2})}, & \text{if } s = 0, \\ \frac{2^{nq-n} \Gamma(\frac{n-1}{2}) \Gamma(\frac{1-n+nq}{2})}{\Gamma(\frac{nq}{2})}, & \text{if } s = 1. \end{cases}$$

## 4. THE PROOF OF THEOREM 1.2

The proof of Theorem 1.2 lies on the following lemmas.

**Lemma 4.1.** *Let  $1 \leq q \leq 2$  and*

$$a_q(r) = \int_0^\pi \sin^{n-2} t (1 + r^2 - 2r \cos t)^{nq/2+n-1} dt.$$

*Then*

$$\max_{0 \leq r \leq 1} a_q(r) = \begin{cases} \frac{\sqrt{\pi} \Gamma(\frac{n-1}{2})}{\Gamma(\frac{n}{2})}, & \text{if } q < 2 - 2/n, \\ \frac{2^{nq-n} \Gamma(\frac{n-1}{2}) \Gamma(\frac{1-n+nq}{2})}{\Gamma(\frac{nq}{2})}, & \text{if } q \geq 2 - 2/n. \end{cases}$$

*Proof of Lemma 4.1.* By using a well-known formula (3.2) for the derivative of Gauss hypergeometric function we obtain

$$\begin{aligned} g(r) &:= \frac{d}{dr} \left( F\left(-1 + n - \frac{nq}{2}, \frac{1}{2}(n - nq), \frac{n}{2}, r^2\right) \right) \\ &= r \cdot (2 + n(-2 + q))(-1 + q) F\left(n - \frac{nq}{2}, \frac{1}{2}(2 + n - nq), \frac{2 + n}{2}, r^2\right). \end{aligned}$$

Let

$$h(r) = F\left(n - \frac{nq}{2}, \frac{1}{2}(2 + n - nq), \frac{2 + n}{2}, r^2\right).$$

Put  $b = n - \frac{nq}{2}$ . As  $q < 2$ , then  $b > 0$ . Further

$$h(r) = F(b, b + 1 - n/2, n/2 + 1, r^2).$$

Because  $n/2 + 1 > b > 0$ , from (3.1) we obtain

$$\begin{aligned} h(r) &= F(b + 1 - n/2, b, n/2 + 1, r^2) \\ &= \frac{\Gamma(n/2 + 1)}{\Gamma(b) \Gamma(n/2 + 1 - b)} \int_0^1 \frac{t^{b-1} (1-t)^{n/2-b}}{(1-tr^2)^{b+1-n/2}} dt. \end{aligned}$$

Hence

$$(4.1) \quad g(r) > 0, \text{ for all } 1 < q < 2 \text{ and } 0 < r < 1.$$

On the other hand if  $q < 2 - 2/n$  then

$$r \cdot (2 + n(-2 + q))(-1 + q) > 0$$

and therefore  $g(r) < 0$ . If  $q > 2 - 2/n$ , then

$$r \cdot (2 + n(-2 + q))(-1 + q) < 0$$

and therefore  $g(r) > 0$ . From (4.1) we obtain

$$\max_{0 \leq r \leq 1} a_q(r) = \begin{cases} a_q(0), & \text{if } q < 2 - 2/n; \\ a_q(1), & \text{if } q \geq 2 - 2/n. \end{cases}$$

From Lemma 3.2 we obtain the conclusion of the lemma.  $\square$

**Lemma 4.2.** *For  $n \geq 2$  and  $q \geq 2$  integrals*

$$\int_0^\pi \sin^{n-2} t (1 + r^2 - 2r \cos t)^{nq/2-n+1} dt$$

*are monotone increasing with respect to the parameter  $r$ ,  $0 \leq r \leq 1$ .*

*Proof.* Let

$$a_q(r) = \int_0^\pi \sin^{n-2} t (1 + r^2 - 2r \cos t)^{nq/2-n+1} dt.$$

For  $0 < r < 1$  we have

$$\begin{aligned} a'_q(r) &= (nq - 2n + 2) \int_0^\pi \sin^{n-2} t (1 + r^2 - 2r \cos t)^{nq/2-n} (r - \cos t) dt \\ &= (nq - 2n + 2)r \int_0^\pi \sin^{n-2} t (1 + r^2 - 2r \cos t)^{nq/2-n} dt \\ &\quad - (nq - 2n + 2) \int_0^\pi \sin^{n-2} t (1 + r^2 - 2r \cos t)^{nq/2-n} \cos t dt \geq 0 \end{aligned}$$

because

$$(4.2) \quad \int_0^\pi \sin^{n-2} t (1 + r^2 - 2r \cos t)^{nq/2-n} dt \geq 0$$

and

$$(4.3) \quad \int_0^\pi \sin^{n-2} t \cos t (1 + r^2 - 2r \cos t)^{nq/2-n} dt \leq 0.$$

The first relation (4.2) follows easily. The integral in (4.3) can be transformed:

$$\begin{aligned} &\int_0^\pi \sin^{n-2} t \cos t (1 + r^2 - 2r \cos t)^{nq/2-n} dt \\ &= - \int_{-\pi/2}^{\pi/2} \cos^{n-2} t \sin t (1 + r^2 + 2r \sin t)^{nq/2-n} dt \\ &= \int_0^{\pi/2} \cos^{n-2} t \sin t ((1 + r^2 - 2r \sin t)^{nq/2-n} - (1 + r^2 + 2r \sin t)^{nq/2-n}) dt. \end{aligned}$$

The sub-integral expression is non-positive and consequently the integral is also non-positive. Since  $a_q(r)$  is monotone increasing on the interval  $(0, 1)$  and continuous on the segment  $[0, 1]$  we have conclusion.  $\square$

*Proof of Theorem 1.2.* By using Lemma 4.1, Lemma 4.2 and (2.1), we have

$$C_p^q = 1$$

if  $q \leq 2 - 2/n$  and

$$C_p^q = \frac{2^{nq-n} \Gamma(\frac{n}{2}) \Gamma(\frac{1-n+nq}{2})}{\sqrt{\pi} \Gamma(\frac{nq}{2})}$$

if  $q > 2 - 2/n$ .  $\square$



*Remark 4.3.* Note that in the case  $n = 3$  we can find very explicit sharp point estimate. Using classical Newton-Leibnitz theorem we have for  $r \neq 0$

$$\begin{aligned} \int_0^\pi \sin t (1 + r^2 - 2r \cos t)^{3q/2-2} dt &= \frac{1}{2r(3q/2-1)} \int_0^\pi d_t (1 + r^2 - 2r \cos t)^{3q/2-1} \\ &= \frac{1}{(3q-2)r} ((1+r)^{3q-2} - (1-r)^{3q-2}) \end{aligned}$$

So for  $x \in B^3, x \neq 0$ , since  $\frac{\Gamma(3/2)}{\sqrt{\pi}} = 1/2$

$$|u(x)| \leq \frac{1}{(1-|x|^2)^{2/p}} \left( \frac{((1+|x|)^{3q-2} - (1-|x|)^{3q-2})}{2(3q-2)|x|} \right)^{1/q} \|u\|_{h^p(B^3)}.$$

For  $x = 0$  we have  $|u(0)| \leq \|u\|_{h^p(B^n)}, n \geq 2$ .

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